On the group theoretical meaning of conformal field theories in the framework of coadjoint orbits

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We present a unifying approach to conformal field theories and other geometric models within the formalism of coadjoint orbits of infinite dimensional Lie groups with central extensions. Starting from the previously obtained general formula for the symplectic action in terms of two fundamental group one-cocycles, we derive the most general form of the Polyakov–Wiegmann composition laws for any geometric model. These composition laws are succinct expressions of all pertinent Noether symmetries. As a basic consequence we obtain Ward identities allowing for the exact quantum solvability of any geometric model.

It has been recognized for the last few years that the whole dynamics of $D=2$ conformal field theories is manifested through the structure of the underlying infinite-dimensional symmetry groups [1]. In the present letter we discuss a general group-theoretical approach based on the method of coadjoint group orbits [2] [#1] and show how to obtain in terms of most general fundamental group-covariant objects a unified formalism for treating classical and quantum properties of $D=2$ conformal field theories and other geometric models.

We first briefly review the basic facts of the formalism. Let G be a Lie group which has a non-trivial central extension $\tilde{G}$. The elements of the corresponding Lie algebra $\tilde{\mathscr{G}}=\mathscr{G}+\mathbb{R}$ are represented by pairs (ξ, n), where n is a central element. Let $\tilde{\mathscr{G}}^*$ with elements (B, c) be the dual of $\tilde{\mathscr{G}}$ relative to a bilinear form

$$\langle (B,c) \mid (\xi, n)\rangle = \langle B|\xi\rangle_0 + cn ,$$

which is an extension of the natural bilinear form $\langle\cdot|\cdot\rangle_0$ on $\mathscr{G}\times\mathscr{G}^*$. The adjoint actions of G and $\mathscr{G}$ on $\mathscr{G}$ are given by $\mathrm{Ad}_0(g)\,\xi=g\xi g^{-1}$ and $\mathrm{ad}_0(\xi_1)\,\xi_2=[\xi_1, \xi_2]$. By duality of $\langle\cdot|\cdot\rangle_0$ these transformations induce the corresponding coadjoint actions $\mathrm{Ad}_0^*(g)$ and $\mathrm{ad}_0^*(\xi)$ on the dual space $\mathscr{G}^*$. Given a one-cocycle S on G with values in $\mathscr{G}^*$ satisfying the following cocycle condition:

$$(\delta S)(g_1, g_2) = \mathrm{Ad}_0^*(g_1)\, S(g_2) - S(g_1 g_2) + S(g_1) = 0 \tag{1}$$

and such that $S(I)=0$, the above adjoint and coadjoint actions can be extended to $\tilde{\mathscr{G}}$ and its dual space as follows:

$$\mathrm{Ad}(g)(\xi, n) = (\mathrm{Ad}_0(g)\,\xi,\, n+\lambda\langle S(g^{-1})|\xi\rangle_0) , \tag{2}$$

$$\mathrm{Ad}^*(g)(B,c) = (\mathrm{Ad}_0^*(g)\, B + c\lambda S(g), c) , \tag{3}$$

where λ is a constant determined by a specific model. The infinitesimal limit of the adjoint representation of the Lie group reproduces the adjoint representation of its Lie algebra,

$$\mathrm{ad}_{(\xi_1, n_1)}(\xi_2, n_2) = [(\xi_1, n_1), (\xi_2, n_2)]$$
$$= ([\xi_1, \xi_2], \omega(\xi_1, \xi_2)) . \tag{4}$$

Consistency with the finite transformation requires

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[#1] For a formulation appropriate for path integral quantization, see refs. [3–5].

that the Lie algebra cocycle in the above formula is defined in terms of the infinitesimal limit of group one-cocycle $s(\epsilon)=S(I+\epsilon)$ as $\omega(\xi, \eta)=-\lambda\langle s(\xi)|\eta\rangle_0$. It is interesting to observe that it is possible to determine uniquely the group one-cocycle $S(g)$ in terms of the Lie algebra cocycle. Applying namely the cocycle condition (1) to $S(g(I+\xi)g^{-1})$ one obtains

$$\omega(\mathrm{Ad}_0(g)\,\xi, \mathrm{Ad}_0(g)\,\eta)-\omega(\xi, \eta)$$
$$=\lambda\langle S(g^{-1})|[\xi, \eta]\rangle_0 . \tag{5}$$

This formula has been obtain by Kirillov [6] for the purpose of determining the group adjoint action from the underlying Lie algebra structure.

For completeness we also list the corresponding Lie algebra version of the coadjoint action: $\mathrm{ad}^*(\xi, n)\,(B, c)=\mathrm{ad}_0^*(\xi)\,B+c\lambda s(\xi), 0)$.

In the physicist's language $\omega(\ ,\ )$ in (4) is just the "anomaly" of the algebra $\mathscr{G}$, while $S(g)$ in (2) and (3) is the "integrated anomaly", i.e. the "anomaly" corresponding to the finite group transformations.

In this framework one can define the Kostant–Kirillov symplectic structure [2] on each coadjoint orbit $\mathcal{O}_{(B_0,c)}$ (i.e. an orbit in the dual space $\tilde{\mathscr{G}}^*$ with a generic point (B_0, c)) in terms of the non-degenerate and closed two-form Ω. Locally the symplectic two-form can be given in terms of the one-form α such that $\Omega=\mathrm{d}\alpha$. Therefore, the corresponding symplectic action, whose phase space is a coadjoint orbit of the special class $\mathcal{O}_{(0,c)}$ [#2], is given by [7]

$$W[g]\equiv\int\alpha=-\lambda c\int\langle\mathscr{S}(g)|\mathscr{Y}(g)\rangle . \tag{6}$$

In (6) the following notations are used. The integral on the RHS is over a one-dimensional curve on the orbit $\mathcal{O}_{(0,c)}$. The pair $\mathscr{S}(g)=(S(g), 1/\lambda)$ is an element of the coadjoint orbit, transforming according to the coadjoint action in (3) as $\mathscr{S}(g_1g_2)=\mathrm{Ad}^*_{g_1}\mathscr{S}(g_2)$ due to the cocycle condition (1). Moreover, $\mathscr{S}(g)$ varies along the co-orbit according to $\mathrm{d}\mathscr{S}(g)=\mathrm{ad}^*(\mathscr{Y}(g))\mathscr{S}(g)$, with $\mathscr{Y}(g)=(y(g), m_{y(g)})$ being a Maurer–Cartan (MC) one-form on $\mathscr{G}$ satisfying the MC equation $\mathrm{d}\mathscr{Y}=\frac{1}{2}[\mathscr{Y}, \mathscr{Y}]$. The MC equation fixes the central element of $\mathscr{Y}$ to $m_y=\frac{1}{2}\mathrm{d}^{-1}\omega(y, y)$ and thus [7]

$$W[g]=-\lambda c\int[\langle S(g)|y(g)\rangle_0$$
$$-\tfrac{1}{2}\mathrm{d}^{-1}(\langle s(y(g))|y(g)\rangle_0)] . \tag{7}$$

The last term on the RHS of (7) is nothing but the generalization of the well known multivalued term in WZNW models.

The generic MC one-form $y(g)$ is of the form $y(g)=\mathrm{d}g\,g^{-1}$ and its transformation property $y(gh)=y(g)+\mathrm{Ad}_0(g)\,y(h)$ makes it to a G one-cocycle with values in $\mathscr{G}$. The identity

$$m_{y(gh)}=m_{y(g)}+m_{y(h)}+\lambda\langle S(g^{-1})|y(h)\rangle_0 , \tag{8}$$

which can be obtained from (5), ensures that $\mathscr{Y}$ is also a $\tilde{\mathrm{G}}$ one-cocycle: $\mathscr{Y}(gh)=\mathscr{Y}(g)+\mathrm{Ad}(g)\,\mathscr{Y}(h)$. The mismatch between transformation properties of $\mathscr{S}$ and $\mathscr{Y}$ leads to the following important composition law valid for all geometric actions (6):

$$W[gh]=-\lambda c\int\langle\mathscr{S}(gh)|\mathscr{Y}(gh)\rangle$$
$$=-\lambda c\int\langle\mathscr{S}(h)|\mathrm{Ad}_{g^{-1}}\mathscr{Y}(\mathrm{gh})\rangle$$
$$=W[g]+W[h]+c\lambda\int\langle S(h)|y(g^{-1})\rangle_0 . \tag{9}$$

The above relation generalizes the Polyakov–Wiegmann composition law for the WZNW action [8] to the arbitrary symplectic actions on coadjoint orbits [7].

Before studying the consequences of (9) it is worth mentioning that by substituting $h=g^{-1}$ into (8) one easily establishes, as a byproduct of (8), an alternative form of the symplectic action (6)

$$\frac{1}{c}W[g]=-\int[m_{y(g)}+\lambda\langle S(g^{-1})|y(g^{-1})\rangle_0]$$
$$=\int m_{y(g^{-1})} , \tag{10}$$

where we used relations $S(g^{-1})=-\mathrm{Ad}_0^*(g^{-1})\,S(g)$ and $y(g^{-1})=-\mathrm{Ad}_0(g^{-1})\,y(g)$ following from the appropriate cocycle conditions. The RHS of (10) is easily recognized as the form of the symplectic action proposed in ref. [4]. Thus, eq. (10) provides the proof of the equivalence between the actions derived in refs. [5] and [4].

Let us stress that the basic relations (9) and (10)

[#2] Most of the physically relevant geometric field theories, such as WZNW models and induced (super) gravity in $D=2$, are of this class.

are valid for the action densities themselves.

Armed with the composition law (9) we can now analyze the symmetry structure associated with the general geometric action (6). Consider first the right group multiplication generated by infinitesimal $h=I+\eta$. We easily obtain the following transformation rule for $W[g]$:

$$\delta^r_\eta W[g] \equiv W[g(I+\eta)] - W[g]$$
$$= c\lambda \int \langle s(\eta) | y(g^{-1}) \rangle_0 , \qquad (11)$$

valid up to second order in η. We made use in deriving (11) of $W[I+\eta] = -\frac{1}{2}c\lambda \int \langle s(\eta) | \mathrm{d}\eta \rangle_0 = \mathrm{O}(\eta^2)$.

We now can easily determine the condition for the invariance of the action, since $\delta^r_\eta W[g]=0$ holds for the elements $\eta \in \mathscr{H}$ – a subalgebra of $\tilde{\mathscr{G}}$ – which satisfy $s(\eta)=0$. For the corresponding right multiplication by the finite group elements h we find directly from (9) that $S(h)=0$ implies $W[gh]-W[g]=0$, and hence the geometric action in (6) remains invariant under the right multiplication with elements $h \in \mathrm{H}$, in the subgroup of G corresponding to $\mathscr{H}$. Clearly H is the isotropy group of the orbit of $(0, c)$ according to (3).

Variation in (11) with an arbitrary η yields the following general equation of motion:

$$s(y(g^{-1}))=0 , \qquad (12)$$

and hence on-shell $y(g^{-1})$ must belong to the stationary subalgebra $\mathscr{H}$. Moreover, from (1) and $\mathrm{d}\mathscr{S}(g) = \mathrm{ad}^*(\mathscr{Y}(g))\, \mathscr{S}(g)$ one finds that

$$s(y(g^{-1})) = -\mathrm{Ad}^*_0(g^{-1})\, \mathrm{d}S(g) . \qquad (13)$$

Correspondingly $S(g)$ must remain constant on shell.

We now turn our attention to the issue of invariance of $W[g]$ under the left group multiplication. Substituting $g=I+\epsilon$ in (9) find the following transformation law:

$$\delta^l_\epsilon W[h] \equiv W[(I+\epsilon)\, h] - W[h]$$
$$= -c\lambda \int \langle S(h) | \mathrm{d}\epsilon \rangle_0 . \qquad (14)$$

This reproduces the Noether theorem for the left multiplication [9] with $S(h)$ being a constant of motion. With $S(h)$ playing the role of generator of the Noether symmetry the Poisson structure is determined by $\delta^l_\epsilon S(h) = \{S(h), \langle S(h) | \epsilon \rangle_0\}$. Since on the other hand we have from (1) $\delta^l_\epsilon S(h) = \mathrm{ad}^*_0(\epsilon)\, S(h) + s(\epsilon)$, we obtain the following Poisson bracket algebra (with arbitrary $\epsilon, \eta \in \mathscr{G}$):

$$\{\langle S(h) | \epsilon \rangle_0, \langle S(h) | \eta \rangle_0\}$$
$$= -\langle \mathrm{ad}^*_0(\epsilon)\, S(h) + s(\epsilon) | \eta \rangle_0$$
$$= \langle S(h) | [\epsilon, \eta] \rangle_0 - \langle s(\epsilon) | \eta \rangle_0 .$$

One easily observes that this Poisson bracket relation for the Noether charge generates the original commutators of the extended Lie algebra $\tilde{\mathscr{G}}$ given in (4).

Finally, let us consider the quantum generating functional for the correlation functions $\langle\langle y_{t_1}(g^{-1}) \ldots y_{t_n}(g^{-1}) \rangle\rangle$ of the group-covariant "composite" fields $y_t(g^{-1})$ (with the notation for the one-form $y(g^{-1}) = \mathrm{d}t\, y_t(g^{-1})$ along a curve on the orbit $\mathscr{O}_{(0,c)}$ with parameter t):

$$Z[j] \equiv \exp(\mathrm{i}\mathscr{W}[j])$$
$$= \int \mathscr{D}g \exp\left[\mathrm{i}\left(W[g] + \int \langle j | y(g^{-1}) \rangle_0\right)\right] . \qquad (15)$$

Performing a change of variables $g \to g(I+\eta)$ in (15) and accounting for the transformation properties of $W[g]$ (11) $y(g^{-1}) - y((I-\eta)\, g^{-1}) = \mathrm{d}\eta - [y(g^{-1}), \eta]$, we obtain the following Ward identity:

$$\partial_t j - \mathrm{ad}^*_0\left(\frac{\delta \mathscr{W}}{\delta j}\right) j - c\lambda s\left(\frac{\delta \mathscr{W}}{\delta j}\right) = 0 . \qquad (16)$$

We note that (16) represents a closed system of equations allowing recursive determination of higher order correlation functions in terms of the lower ones. For the lowest non-trivial correlation function one gets

$$\mathrm{i}c\lambda s^{(1)} \langle\langle y_t(g^{-1}) \otimes y_{t'}(g^{-1}) \rangle\rangle$$
$$- \partial_t \delta(t-t')\, I \otimes I = 0 . \qquad (17)$$

Here $s^{(1)}$ indicates the action of the infinitesimal cocycle s on the first member in the tensor product $\mathscr{G} \otimes \mathscr{G}$.

Using the infinitesimal versions the generalized composition law given in (11) and (14) we obtain the basic relations

$$\frac{\delta W[\tilde{g}]}{\delta S(\tilde{g})} = -c\lambda y_t(\tilde{g}) ,$$

$$\frac{\delta W[\tilde{g}]}{\delta y_t(\tilde{g}^{-1})} = -c\lambda S(\tilde{g}^{-1}) . \tag{18}$$

Now parametrizing the source $j \in \tilde{\mathscr{G}}$ in (15), (16) as $j \equiv c\lambda S(\tilde{g})$ for some $\tilde{g} \in G$, and using the first relation in (18), we can write the explicit solution of the Ward identities (16) in the form

$$\mathscr{W}[j \equiv c\lambda S(\tilde{g})] = -W[\tilde{g}] . \tag{19}$$

Similarly, for the quantum effective action $\Gamma[y] = \mathscr{W}[j] - \int \langle j | \delta\mathscr{W}/\delta j \rangle_0$ with $y \equiv \delta\mathscr{W}/\delta j$, we easily get by using (18), (19) and (9),

$$\Gamma[y \equiv y_t(\tilde{g})] = W[\tilde{g}] . \tag{20}$$

In the particular case of $G = \mathrm{Diff}\, S^1$, corresponding to the Virasoro algebra, the elements $g \in \mathrm{Diff}\, S^1$ are smooth diffeomorphisms $F: x \to F(x)$. One finds that the one-cocycle $S(g) = S(F)$ is the schwarzian of F. Furthermore one makes the following connection between the general geometric objects and their specific realization for $G = \mathrm{Diff}\, S^1$: $(t, x) \equiv (x^+, x^-)$; $s(\xi) = \partial_-^3 \xi$; $\mathrm{ad}_0^*(\xi)\, \eta = \xi \partial_- \eta + 2(\partial_- \xi)\, \eta$. If we also let g^{-1} be described by $x \to f(x)$ then $y_t(g^{-1}) = \partial_+ f/\partial_- f = h_{++}$ using Polyakov's notation [10] (see also ref. [5]). With these identifications eqs. (9), (13) and (12) are easily recognized as, respectively, Polyakov's composition law, the "anomaly" equation and the equation of motion for induced $D=2$ gravity [10]. Similarly, (16) and (17) reduce to the Ward identities for $\langle\langle h_{++} ... h_{++} \rangle\rangle$ and the equation explicitly determining the two-point function $\langle\langle h_{++} h_{++} \rangle\rangle$ in ref. [10].

In the above solution of the above Ward identities in (16), (19) and (20) we had not taken into account the non-invariance (in general) of the integration measure $\mathscr{D}g$ in (15) under the right group translation $g \to g(I+\eta)$. The corresponding jacobian is proportional to the original geometric action $W[g]$ (6), i.e. it gives rise to a renormalization of the central charge parameter c. In the case of the Virasoro group and its (1, 0) super-extension, renormalization of c was found in refs. [10,11].

Derivations, similar to the one leading to (16), yield Ward identities for correlation functions involving arbitrary "composite" fields which have covariant group properties (e.g. the cocycle $S(g)$). These Ward identities ((16) and their analogues) provide the basis for the exact quantum solvability of any geometric field theory. The natural further step is to exploit the implications of the general composition law (9) and the group-covariant equation of motion (12) to study the quantum renormalization properties and obtain spectra of anomalous dimensions, in a model-independent way.

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Note added. After completion of the manuscript we became aware of a preprint [12] where the group composition law for $D=2$ induced (1, 0) supergravity is derived. This result, as well as its generalizations to (2, 0) and (3, 0) induced supergravity, can be straightforwardly obtained (see ref. [13]) as particular cases of the general composition law (eq. (9) above) since both $S(g)$ [14] and $y(g)$ [7,15,16] are explicitly known.

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